New College Bradford

A-Level Further Mathematics

Y12-13

Summer Independent Learning

	Content	Answers
Consolidation of A level-only content	Oblique asymptotes Complex powers and roots Further vectors Further matrices Yates' correction	Oblique asymptotes Complex powers and roots Further vectors Further matrices Yates' correction
Review of AS content	Questions	Markschemes (2021 November Papers)

As usual – any problems, email me!

Consolidation: Oblique asymptotes

Knowledge Check	
1.	Find the equations of the asymptotes of: $y=\frac{2x^2}{1-x}$
2.	Sketch a fully-labelled graph of: $y=\frac{2x^2-1}{2x+3}$
3.	Find the equations of the asymptotes and hence sketch: $y=\frac{x^3-x}{x^2-4}$ *(Bonus points if you can find and classify the stationary points! You'll need to use calculus – once you've got the sketch, can you see why the FM method using intersections with $y=k$ won't work here?)*

Reasoning

A curve with equation $y=\frac{ax^2+bx-1}{x+1}$ has an asymptote $y=4x-2$

a Find the values of a and b

b Write down the equation of the other asymptote.

c Without using calculus, find the coordinates of the turning points.

d Sketch the curve.

Consolidation: Complex powers and roots

Knowledge Check	
1.	Write in exponential form: $-4-8i$ $\quad 4\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)$ $\quad 3\left(\cos\left(\frac{5\pi}{6}\right)-i\sin\left(\frac{5\pi}{6}\right)\right)$ Write in the form $a+bi$: $7e^{-\frac{\pi}{3}i}$ Given that $z=5e^{\frac{2\pi}{7}i}$ and $w=\frac{1}{5}e^{-\frac{\pi}{7}i}$, calculate the value of **a** $\|zw\|$ **b** $\left\|\frac{z}{w}\right\|$ **c** $\arg(zw)$ **d** $\arg\left(\frac{z}{w}\right)$
2.	Given that $z=4\left(\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right)$, express in exact Cartesian form **a** z^2 **b** z^3 **c** $\frac{1}{z}$ **d** $16z^{-4}$ Given that $z=-\sqrt{3}+i$, use de Moivre's theorem to write the following in Cartesian form. **a** z^4 **b** z^{-3} **c** z^{-2} **d** $\frac{8}{z^6}$
3.	Solve each of these equations, giving your solutions in exponential form **a** $z^3=4\sqrt{2}+4\sqrt{2}i$ **b** $z^3=-4\sqrt{2}+4\sqrt{2}i$ **c** $z^3=-4\sqrt{2}-4\sqrt{2}i$ **d** $z^3=4\sqrt{2}-4\sqrt{2}i$

Reasoning
A complex number z has modulus 1 and argument θ **a** Show that $z^n+\frac{1}{z^n}=2\cos(n\theta)$ **b** Show that $z^n-\frac{1}{z^n}=2i\sin(n\theta)$ Given that ω is a complex cube root of unity **a** Show that $1+\omega+\omega^2=0$ **b** Evaluate the following expressions. **i** $(1+\omega)^2-\omega$ **ii** $(1+\omega)(1+\omega^2)$ **iii** $\omega(\omega+1)$ **iv** $\frac{2\omega+1}{\omega-1}+\omega$

Challenge
The points A, B and C represent the solutions to the equation $z^3=-27i$ **a** Find the solutions to the equation in the form $a+bi$ **b** Calculate the exact **i** Area, **ii** Perimeter of triangle ABC

Consolidation: Further vectors

Knowledge Check

1.

Use the distributive and anticommutative properties of the vector product to show that

a $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{c}=(\mathbf{b}+\mathbf{a})\times(\mathbf{c}-\mathbf{a})$

b $\mathbf{b}\times(2\mathbf{a}+\mathbf{c})-\mathbf{c}\times(\mathbf{b}-\mathbf{c})=2\mathbf{b}\times(\mathbf{a}+\mathbf{c})$

Find a unit vector which is perpendicular to both $\begin{pmatrix}-5\\0\\0\end{pmatrix}$ and $\begin{pmatrix}-2\\1\\-3\end{pmatrix}$

Find the values of a, b and c given that

$$\begin{pmatrix}3\\-4\\a\end{pmatrix}\times\begin{pmatrix}0\\2\\b\end{pmatrix}=\begin{pmatrix}10\\9\\c\end{pmatrix}$$

Calculate the exact area of the triangles with vertices at $(1, 1, -2)$, $(0, 1, -1)$ and $(-2, 0, 1)$

2.

A plane contains the points $(5, 1, 1)$, $(-2, 0, 0)$ and $(6, 2, -5)$

Find the equation of the plane in

a Vector form, **b** Scalar product form,

c Cartesian form.

Calculate the acute angle between the line $\mathbf{r}=2\mathbf{i}-8\mathbf{j}+\mathbf{k}+t(\mathbf{j}+\mathbf{k})$ and the plane $\mathbf{r}\cdot(6\mathbf{i}-\mathbf{j}+5\mathbf{k})=9$

3.

Find the shortest distance between each point and plane.

a $(2, 5, 1)$ and $2x-4y+4z+5=0$

Find the point of intersection of the line with equation $\dfrac{x-3}{4}=\dfrac{y-2}{5}=\dfrac{z-1}{-3}$ and the plane with equation $\mathbf{r}=13\mathbf{i}+7\mathbf{j}-6\mathbf{k}+s(\mathbf{i}+2\mathbf{j}-5\mathbf{k})+t(-\mathbf{i}+\mathbf{j}+2\mathbf{k})$

Reasoning

The line l passes through the points $A(1, 4, -2)$ and $B(0, 2, -7)$

The plane Π has equation $5x-5y+z=3$

a Find the shortest distance from the plane to the point

i A **ii** B

b Does the line intersect the plane? Explain your answer.

Do these equations represent the same line? Explain how you know.

A: $\dfrac{x-3}{-2}=\dfrac{y+1}{4}=\dfrac{z-5}{-6}$

B: $\left(\mathbf{r}-\begin{pmatrix}4\\-3\\8\end{pmatrix}\right)\times\begin{pmatrix}-3\\6\\-9\end{pmatrix}=\mathbf{0}$

Challenge

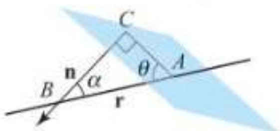
Three lines have equations as follows:

L_1: $\mathbf{r}=6\mathbf{i}-3\mathbf{j}+\lambda(\mathbf{i}+\mathbf{k})$,

L_2: $\mathbf{r}=s(3\mathbf{i}-\mathbf{j}+\mathbf{k})$ and

L_3: $\mathbf{r}=t(\mathbf{j}+2\mathbf{k})$

The lines L_1 and L_2 intersect at the point A, L_1 and L_3 intersect at the point B and L_2 and L_3 intersect at the point C, as shown.



Calculate the exact area of triangle ABC

Consolidation: Further matrices

	Knowledge Check
1.	Find the determinant of: $\begin{pmatrix} 2a & 3 & a \\ -b & 0 & b \\ 3 & 5 & 1 \end{pmatrix}$ Find the inverse of: $\begin{pmatrix} -4a & 2 & 1 \\ a & 0 & 2 \\ 3a & -1 & 1 \end{pmatrix}$ **a** Use one or more column operations to write $\begin{vmatrix} 2 & 1 & -3 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ as a determinant with at least two zero elements. **b** Hence find $\det\begin{pmatrix} 2 & 1 & -3 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ Use row or column operations to show that **a** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(ab+ac+bc-a^2-b^2-c^2)$ **b** $\begin{vmatrix} a^2 & b^2 & c^2 \\ bc & ca & ab \\ 1 & 1 & 1 \end{vmatrix} = (b-a)(c-a)(c-b)(a+b+c)$ **c** $\begin{vmatrix} a+b & a+c & b+c \\ c & b & a \\ c^2 & b^2 & a^2 \end{vmatrix} = (b-a)(a-c)(c-b)(a+b+c)$ **d** $\begin{vmatrix} a & -b & c \\ c-b & a+c & a-b \\ -bc & ac & -ab \end{vmatrix} = (a+b)(c-a)(b+c)(a-b+c)$
2.	Describe fully the geometric arrangements of the following systems of planes: $x+y-2z=3$, $2x-3y+5z=4$, $5x+2y+z=-3$ $3x-2y+z=7$, $6x-4y+2z=5$, $x+3y+2z=3$ $x+2z=1$, $-2x+y+4z=0$, $9x-2y+2z=5$
3.	Find the eigenvalues and eigenvectors of: $\begin{pmatrix} 1 & 0 & -5 \\ 4 & 5 & 1 \\ 0 & 0 & -3 \end{pmatrix}$ Given $\mathbf{B}=\begin{pmatrix} 2 & 6 \\ 1 & -3 \end{pmatrix}$, write $\mathbf{B}$ in the form $\mathbf{B}=\mathbf{PDP}^{-1}$
	Reasoning
	Given that the determinant of $\begin{pmatrix} a & 2 & -3 \\ 1 & a & 4 \\ 3 & 2 & 5 \end{pmatrix}$ is 128, calculate each of these determinants. Justify your answer in each case. **a** $\begin{vmatrix} 1 & a & 4 \\ a & 2 & -3 \\ 3 & 2 & 5 \end{vmatrix}$ **e** $\begin{vmatrix} a & 6 & -3 \\ 1 & 3a & 4 \\ 3 & 6 & 5 \end{vmatrix}$ **c** $\begin{vmatrix} a+1 & 2+a & 1 \\ 1 & a & 4 \\ 3 & 2 & 5 \end{vmatrix}$ **g** $\begin{vmatrix} -3 & 2a & 2 \\ 4 & 2 & a \\ 5 & 6 & 2 \end{vmatrix}$
	Challenge
	Given that $\mathbf{T}=\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ **a** Work out $\mathbf{T}^5$ **b** Show that $\mathbf{T}^n = 5^{n-1}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

Consolidation: Yates' correction

Knowledge Check	
1.	Write down the formula for Yates' correction and the condition for when you would apply it
2.	The lunch choices of two year groups at a school are tested for association. A sample of 63 students in year 12 and year 13 are asked whether they prefer fish 'n' chips or salad. **a** Draw a table of expected frequencies (correct to 3 sf). **b** Draw a table of X^2 contributions and calculate the X^2 test statistic. **c** At the 10% level the critical value is 2.71. Determine the conclusion of the test.

Review: MEI AS Further Maths 2021 Papers

Core pure

1 Using standard summation formulae, find $\sum_{r=1}^{n}(r^2-3r)$, giving your answer in fully factorised form. **[3]**

2 The equation $3x^2-4x+2=0$ has roots α and β.

Find an equation with integer coefficients whose roots are $3-2\alpha$ and $3-2\beta$. **[3]**

3 Three planes have the following equations.

$$\begin{aligned} 2x-3y+\ z &= -3, \\ x-4y+2z &= 1, \\ -3x-2y+3z &= 14. \end{aligned}$$

(a) **(i)** Write the system of equations in matrix form. **[1]**

(ii) Hence find the point of intersection of the planes. **[2]**

(b) **In this question you must show detailed reasoning.**

Find the acute angle between the planes $2x-3y+z=-3$ and $x-4y+2z=1$. **[4]**

4 Anika thinks that, for two square matrices $\mathbf{A}$ and $\mathbf{B}$, the inverse of $\mathbf{AB}$ is $\mathbf{A}^{-1}\mathbf{B}^{-1}$. Her attempted proof of this is as follows.

$$\begin{aligned} (\mathbf{AB})(\mathbf{A}^{-1}\mathbf{B}^{-1}) &= \mathbf{A}(\mathbf{BA}^{-1})\mathbf{B}^{-1} \\ &= \mathbf{A}(\mathbf{A}^{-1}\mathbf{B})\mathbf{B}^{-1} \\ &= (\mathbf{AA}^{-1})(\mathbf{BB}^{-1}) \\ &= \mathbf{I}\times\mathbf{I} \\ &= \mathbf{I} \end{aligned}$$

Hence $(\mathbf{AB})^{-1}=\mathbf{A}^{-1}\mathbf{B}^{-1}$

(a) Explain the error in Anika's working. **[2]**

(b) State the correct inverse of the matrix $\mathbf{AB}$ and amend Anika's working to prove this. **[3]**

5 Prove by induction that $\sum_{r=1}^{n} r\times 2^{r-1}=1+(n-1)2^n$ for all positive integers n. **[5]**

6 A transformation T of the plane has associated matrix $\mathbf{M} = \begin{pmatrix} 1 & \lambda+1 \\ \lambda-1 & -1 \end{pmatrix}$, where λ is a non-zero constant.

(a) **(i)** Show that T reverses orientation. **[3]**

(ii) State, in terms of λ, the area scale factor of T. **[1]**

(b) **(i)** Show that $\mathbf{M}^2 - \lambda^2\mathbf{I} = \mathbf{0}$. **[2]**

(ii) Hence specify the transformation equivalent to two applications of T. **[1]**

(c) In the case where $\lambda = 1$, T is equivalent to a transformation S followed by a reflection in the x-axis.

(i) Determine the matrix associated with S. **[3]**

(ii) Hence describe the transformation S. **[2]**

7 **(a)** **(i)** Find the modulus and argument of z_1, where $z_1 = 1 + \mathrm{i}$. **[2]**

(ii) Given that $|z_2| = 2$ and $\arg(z_2) = \frac{1}{6}\pi$, express z_2 in $a + b\mathrm{i}$ form, where a and b are exact real numbers. **[2]**

(b) Using these results, find the exact value of $\sin\frac{5}{12}\pi$, giving the answer in the form $\frac{\sqrt{m}+\sqrt{n}}{p}$, where m, n and p are integers. **[5]**

8 **In this question you must show detailed reasoning.**

The equation $x^3 + kx^2 + 15x - 25 = 0$ has roots α, β and $\frac{\alpha}{\beta}$. Given that $\alpha > 0$, find, in any order,

- the roots of the equation,
- the value of k. **[7]**

9 **(a)** On a single Argand diagram, sketch the loci defined by

- $\arg(z-2) = \frac{3}{4}\pi$,
- $|z| = |z + 2 - \mathrm{i}|$. **[4]**

(b) **In this question you must show detailed reasoning.**

The point of intersection of the two loci in part **(a)** represents the complex number w.

Find w, giving your answer in exact form. **[5]**

1 The *specific energy* of a substance has SI unit $\mathrm{J\,kg^{-1}}$ (joule per kilogram).

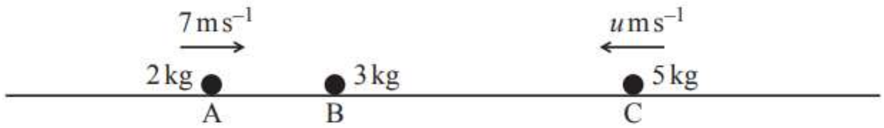
(a) Determine the dimensions of specific energy. **[2]**

A particular brand of protein powder contains approximately 345 Calories (Cal) per 4 ounce (oz) serving. An athlete is recommended to take 40 grams of the powder each day.

You are given that $1\,\mathrm{oz} = 28.35$ grams and $1\,\mathrm{Cal} = 4184\,\mathrm{J}$.

(b) Determine, in joules, the amount of energy in the athlete's recommended daily serving of the protein powder. **[2]**

3 Three small uniform spheres A, B and C have masses $2\,\mathrm{kg}$, $3\,\mathrm{kg}$ and $5\,\mathrm{kg}$ respectively. The spheres move in the same straight line on a smooth horizontal table, with B between A and C. Sphere A moves towards B with speed $7\,\mathrm{m\,s^{-1}}$, B is at rest and C moves towards B with speed $u\,\mathrm{m\,s^{-1}}$, as shown in the diagram.



Spheres A and B collide. Collisions between A and B can be modelled as perfectly elastic.

(a) Determine the magnitude of the impulse of A on B in this collision. **[5]**

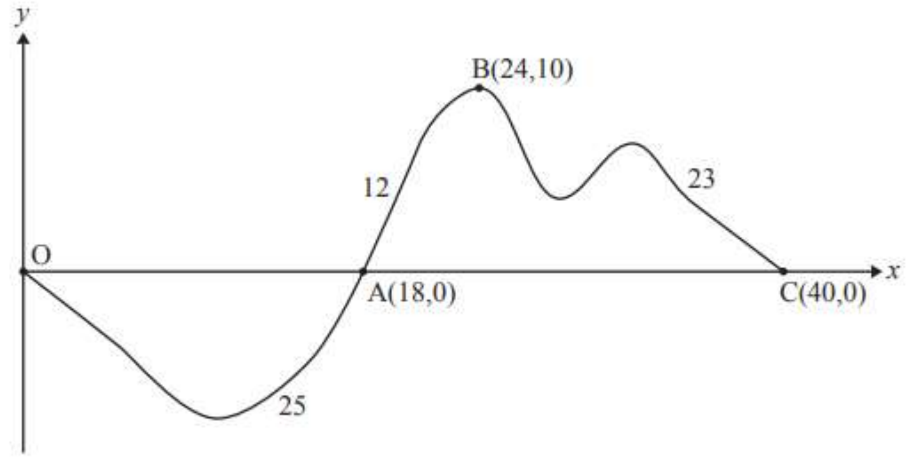
(b) Use this collision to verify that in a perfectly elastic collision no kinetic energy is lost. **[1]**

After the collision between A and B, sphere B subsequently collides with C. The coefficient of restitution between B and C is $\frac{1}{4}$.

(c) Show that, after the collision between B and C, B has a speed of $(1.225 - 0.78125u)\,\mathrm{m\,s^{-1}}$ towards C. **[4]**

(d) Determine the range of values for u for there to be a second collision between A and B. **[2]**

4 The diagram shows the path of a particle P of mass 2 kg as it moves from the origin O to C via A and B. The lengths of the sections OA, AB and BC are given in the diagram. The units of the axes are metres.



P, starting from O, moves along the path indicated in the diagram to C under the action of a constant force of magnitude T N acting in the positive x-direction. As P moves, it does R J of work for every metre travelled against resistances to motion.

It is given that

- the speed of P at O is $3\,\mathrm{m\,s^{-1}}$,
- the speed of P at A is $11\,\mathrm{m\,s^{-1}}$,
- the speed of P at C is $15\,\mathrm{m\,s^{-1}}$.

You should assume that both x- and y-axes lie in a horizontal plane.

(a) By considering the entire path of P from O to C, show that

$$20T - 30R = 108.$$ **[2]**

(b) By formulating a second equation, determine the values of T and R. **[3]**

(c) It is now given that the x-axis is horizontal, and the y-axis is directed vertically upwards. By considering the kinetic energy of P at B, show that the motion as described above is impossible. **[3]**

Statistics a

1 The random variable X represents the clutch size (the number of eggs laid) by female birds of a particular species. The probability distribution of X is given in the table.

r	2	3	4	5	6	7
$\mathrm{P}(X = r)$	0.03	0.07	0.27	0.49	0.13	0.01

(a) Find each of the following.

- $\mathrm{E}(X)$
- $\mathrm{Var}(X)$ **[2]**

On average 65% of eggs laid result in a young bird successfully leaving the nest.

(b) **(i)** Find the mean number of young birds that successfully leave the nest. **[1]**

(ii) Find the standard deviation of the number of young birds that successfully leave the nest. **[1]**

4 It is known that in an electronic circuit, the number of electrons passing per nanosecond can be modelled by a Poisson distribution. In a particular electronic circuit, the mean number of electrons passing per nanosecond is 12.

(a) **(i)** Determine the probability that there are more than 15 electrons passing in a randomly selected nanosecond. **[2]**

(ii) Determine the probability that there are fewer than 50 electrons passing in a randomly selected period of 5 nanoseconds. **[2]**

(b) Explain what you can deduce about the electrons passing in the circuit from the fact that a Poisson distribution is a suitable model. **[2]**

5 A fair spinner has five faces, labelled 0, 1, 2, 3, 4.

(a) State the distribution of the score when the spinner is spun once. **[2]**

(b) Determine the probability that, when the spinner is spun twice, one of the scores is less than 2 and the other is at least 2. **[2]**

(c) Find the variance of the total score when the spinner is spun 5 times. **[3]**

Mechanics b

1 The end O of a light elastic string OA is attached to a fixed point.

Fiona attaches a mass of 1 kg to the string at A. The system hangs vertically in equilibrium and the length of the stretched string is 70 cm.

Fiona removes the 1 kg mass and attaches a mass of 2 kg to the string at A. The system hangs vertically in equilibrium and the length of the stretched string is now 80 cm.

Fiona then removes the 2 kg mass and attaches a mass of 5 kg to the string at A. The system hangs vertically in equilibrium.

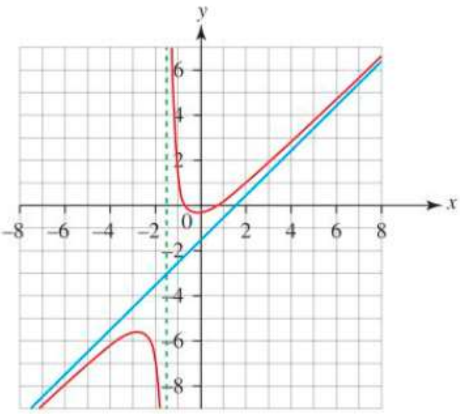
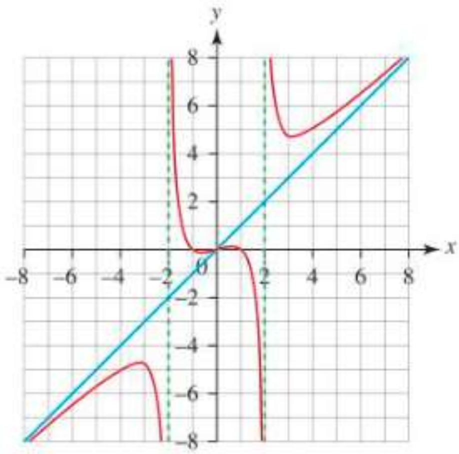
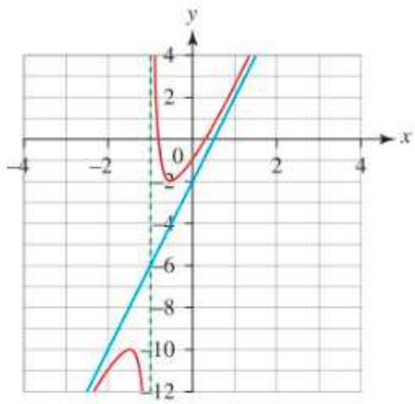
(a) Use the information given in the question to determine expected values for

- the length of the stretched string when the 5 kg mass is attached,
- the elastic potential energy stored in the string in this case. **[7]**

Fiona discovers that, when the mass of 5 kg is attached to the string at A, the length of the stretched string is greater than the expected length.

(b) Suggest a reason why this has happened. **[1]**

CONSOLIDATION ANSWERS

	Oblique asymptotes	Complex powers and roots
1	$y=-2x-2$ and $x=1$	$4\sqrt{5}e^{-i2.03}$ $\quad 4e^{-\frac{2\pi}{3}i}$ $\quad 3e^{-\frac{5\pi}{6}i}$ $\frac{7}{2}-\frac{7\sqrt{3}}{2}i$ **a** 1 **b** 25 **c** $\frac{\pi}{7}$ **d** $\frac{3\pi}{7}$
2	**b** Asymptotes at $x=-\frac{3}{2}$ and $y=x-\frac{3}{2}$ 	**a** -16 **b** $64i$ **c** $\frac{1}{4}i$ **d** $\frac{1}{16}$ **a** $8-8\sqrt{3}i$ **b** $-\frac{1}{8}i$ **c** $\frac{1}{8}+\frac{\sqrt{3}}{8}i$ **d** $-\frac{1}{8}$
3	Asymptotes at $x=\pm 2$ and $y=x$ 	**a** $2e^{\frac{\pi}{12}i}, 2e^{\frac{3\pi}{4}i}, 2e^{-\frac{7\pi}{12}i}$ **b** $2e^{\frac{\pi}{4}i}, 2e^{\frac{11\pi}{12}i}, 2e^{-\frac{5\pi}{12}i}$ **c** $2e^{-\frac{\pi}{4}i}, 2e^{\frac{5\pi}{12}i}, 2e^{-\frac{11\pi}{12}i}$ **d** $2e^{-\frac{\pi}{12}i}, 2e^{\frac{7\pi}{12}i}, 2e^{-\frac{3\pi}{4}i}$
R	**a** $a=4, b=2$ **b** $x=-1$ **c** $\left(-\frac{1}{2},-2\right),\left(-\frac{3}{2},-10\right)$ **d** 	$z=e^{\theta i}$ **a** $z^n+\frac{1}{z^n}=(e^{\theta i})^n+\frac{1}{(e^{\theta i})^n}$ $=e^{n\theta i}+e^{-n\theta i}$ $=2\cos(n\theta)$ as required **b** $z^n-\frac{1}{z^n}=(e^{\theta i})^n-\frac{1}{(e^{\theta i})^n}$ $=e^{n\theta i}-e^{-n\theta i}$ $=2i\sin(n\theta)$ as required **a** $1+\omega+\omega^2=\frac{(1-\omega^3)}{1-\omega}$ $=\frac{(1-1)}{1-\omega}=0$ **b** **i** 0 **ii** 1 **iii** -1 **iv** 0
C		**a** $\frac{3\sqrt{3}}{2}-\frac{3}{2}i, 3i, -\frac{3\sqrt{3}}{2}-\frac{3}{2}i$ **b** **i** $\frac{27\sqrt{3}}{4}$ square units **ii** $3\sqrt{3}(\sqrt{2}+1)$ units

<table>
<tr><th></th><th>Further vectors</th><th>Further matrices</th></tr>
<tr><td>1</td><td>

a $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{c}=-\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}+\mathbf{a}\times\mathbf{c}$

$=\mathbf{b}\times(\mathbf{c}-\mathbf{a})+\mathbf{a}\times\mathbf{c}$

$=(\mathbf{b}+\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ (since $\mathbf{a}\times\mathbf{a}=0$)

b $\mathbf{b}\times(2\mathbf{a}+\mathbf{c})-\mathbf{c}\times(\mathbf{b}-\mathbf{c})=\mathbf{b}\times(2\mathbf{a})+\mathbf{b}\times\mathbf{c}-\mathbf{c}\times\mathbf{b}-\mathbf{c}\times(-\mathbf{c})$

$=2\mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{c}$

$=2\mathbf{b}\times\mathbf{a}+2\mathbf{b}\times\mathbf{c}$

$=2\mathbf{b}\times(\mathbf{a}+\mathbf{c})$

$$\begin{pmatrix} 0 \\ -\frac{3}{\sqrt{10}} \\ -\frac{1}{\sqrt{10}} \end{pmatrix}$$

$b=-3$

$a=1$

$c=6$

$$\frac{\sqrt{2}}{2}$$

</td><td>

$12b-15ab$

$$\begin{pmatrix} \frac{2}{a} & -\frac{3}{a} & \frac{4}{a} \\ 5 & -7 & 9 \\ -1 & 2 & -2 \end{pmatrix}$$

a For example, $\begin{vmatrix} 2 & 1 & -3 \\ 4 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}=\begin{vmatrix} 0 & 1 & -3 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{vmatrix}$ C1 − 2C2

b −7

a $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=\begin{vmatrix} a+b & b+c & c+a \\ b & c & a \\ c & a & b \end{vmatrix}$ R1 + R2

$$=\begin{vmatrix} a+b+c & b+c+a & c+a+b \\ b & c & a \\ c & a & b \end{vmatrix} \text{ R1+R3}$$

$$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$=(a+b+c)[(bc-a^2)-(b^2-ac)+(ab-c^2)]$$

$$=(a+b+c)(ab+ac+bc-a^2-b^2-c^2)$$

as required

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ bc & ca & ab \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} a^2 & b^2-a^2 & c^2 \\ bc & ca-bc & ab \\ 1 & 0 & 1 \end{vmatrix} \text{ C2−C1}$$

$$=\begin{vmatrix} a^2 & b^2-a^2 & c^2-a^2 \\ bc & ca-bc & ab-bc \\ 1 & 0 & 0 \end{vmatrix} \text{ C3−C1}$$

$$=\begin{vmatrix} a^2 & (b-a)(b+a) & c^2-a^2 \\ bc & -c(b-a) & ab-bc \\ 1 & 0 & 0 \end{vmatrix}$$

$$=(b-a)\begin{vmatrix} a^2 & b+a & c^2-a^2 \\ bc & -c & ab-bc \\ 1 & 0 & 0 \end{vmatrix}$$

$$=(b-a)\begin{vmatrix} a^2 & b+a & (c-a)(c+a) \\ bc & -c & -b(c-a) \\ 1 & 0 & 0 \end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix} a^2 & b+a & c+a \\ bc & -c & -b \\ 1 & 0 & 0 \end{vmatrix}$$

$$=(b-a)(c-a)[-b(b+a)--c(c+a)]$$

$$=(b-a)(c-a)(-b^2-ab+c^2+ac)$$

$$=(b-a)(c-a)(c-b)(a+b+c) \text{ as required}$$

$$\begin{vmatrix} a+b & a+c & b+c \\ c & b & a \\ c^2 & b^2 & a^2 \end{vmatrix}=\begin{vmatrix} a+b+c & a+c+b & b+c+a \\ c & b & a \\ c^2 & b^2 & a^2 \end{vmatrix} \text{ R1+R2}$$

$$=(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c & b & a \\ c^2 & b^2 & a^2 \end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix} 0 & 1 & 1 \\ c-b & b & a \\ c^2-b^2 & b^2 & a^2 \end{vmatrix} \text{ C1−C2}$$

$$=(a+b+c)(c-b)\begin{vmatrix} 0 & 1 & 1 \\ 1 & b & a \\ c+b & b^2 & a^2 \end{vmatrix}$$

</td></tr>
</table>

		$=(a+b+c)(c-b)\begin{vmatrix}0&0&1\\1&b-a&a\\c+b&b^2-a^2&a^2\end{vmatrix}$ C2 − C3 $=(a+b+c)(c-b)(b-a)[(a+b)-(c+b)]$ $=(a+b+c)(c-b)(b-a)(a-c)$ as required **d** $\begin{vmatrix}a&-b&c\\c-b&a+c&a-b\\-bc&ac&-ab\end{vmatrix}=\begin{vmatrix}a+b&-b&c\\-b-a&a+c&a-b\\-bc-ac&ac&-ab\end{vmatrix}$ C1 − C2 $=\begin{vmatrix}a+b&-b&c\\-(a+b)&a+c&a-b\\-c(a+b)&ac&-ab\end{vmatrix}$ $=(a+b)\begin{vmatrix}1&-b&b+c\\-1&a+c&-b-c\\-c&ac&-ab-ac\end{vmatrix}$ C3 − C2 $=(a+b)\begin{vmatrix}1&-b&b+c\\-1&a+c&-(b+c)\\-c&ac&-a(b+c)\end{vmatrix}$ $=(a+b)(b+c)\begin{vmatrix}1&-b&1\\-1&a+c&-1\\-c&ac&-a\end{vmatrix}$ $=(a+b)(b+c)\begin{vmatrix}0&-b&1\\0&a+c&-1\\-c+a&ac&-a\end{vmatrix}$ C1 − C3 $=(a+b)(b+c)(c-a)\begin{vmatrix}0&-b&1\\0&a+c&-1\\-1&ac&-a\end{vmatrix}$ $=(a+b)(b+c)(c-a)[-1(b-(a+c))]$ $=(a+b)(b+c)(c-a)(a-b+c)$ as required
2	**a** $\mathbf{r}=\begin{pmatrix}-2\\0\\0\end{pmatrix}+s\begin{pmatrix}7\\1\\1\end{pmatrix}+t\begin{pmatrix}8\\2\\-5\end{pmatrix}$ **b** $\mathbf{n}=\begin{pmatrix}7\\1\\1\end{pmatrix}\times\begin{pmatrix}8\\2\\-5\end{pmatrix}=\begin{pmatrix}-7\\43\\6\end{pmatrix}$ $\begin{pmatrix}-2\\0\\0\end{pmatrix}\cdot\begin{pmatrix}-7\\43\\6\end{pmatrix}=14$ $\mathbf{r}\cdot(-7\mathbf{i}+43\mathbf{j}+6\mathbf{k})=14$ **c** $-7x+43y+6z-14=0$ $21.1°$	Intersect at $(2,-5,-3)$ If we multiply the first equation by 2 we get $6x-4y+2z=14$ so this plane is parallel to the second as they are the same except the constant term. Therefore, the three planes do not meet. $\det\begin{pmatrix}1&0&2\\-2&1&4\\9&-2&2\end{pmatrix}=1(12--8)+2(4-9)$ $=0$ so not a unique solution First equation gives $x=1-2z$ Substitute into second equation to give $-2(1-2z)+y+4z=0\Rightarrow y=2-8z$ Check in third equation: $9(1-2z)+2(2-8z)+2z=5$ So they form a sheaf (the line has equations such as $x=1-2z,\ y=2-8z$)
3	$\frac{7}{6}$ units $(11, 12, -5)$	$\lambda=-3,\ 1,\ 5$, corresponding eigenvectors are $\begin{pmatrix}5\\-3\\4\end{pmatrix},\begin{pmatrix}1\\-1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}$ $\mathbf{B}=\begin{pmatrix}-1&6\\1&1\end{pmatrix}\begin{pmatrix}-4&0\\0&3\end{pmatrix}\begin{pmatrix}-\frac{1}{7}&\frac{6}{7}\\\frac{1}{7}&\frac{1}{7}\end{pmatrix}$

R	**a** **i** $\frac{-20\sqrt{51}}{51}$ **ii** $\frac{-20\sqrt{51}}{51}$ **b** Since A and B are the same distance from Π and are the same side of Π the line *l* must be parallel to the plane therefore it doesn't intersect it. $\begin{pmatrix} -3 \\ 6 \\ -9 \end{pmatrix} = \frac{3}{2}\begin{pmatrix} -2 \\ 4 \\ -6 \end{pmatrix}$ so they are parallel (4, −3, 8) satisfies B so check A: $\frac{4-3}{-2} = -\frac{1}{2}$ $\frac{-3+1}{4} = -\frac{1}{2}$ $\frac{8-5}{-6} = -\frac{1}{2}$ so (4, −3, 8) satisfies both equations therefore they represent the same line.	**a** −128 since rows 1 and 2 swapped **c** 128 since row 2 added to row 1 **e** $128 \times 3 = 384$ since column 2 multiplied by 3 **g** 256 since columns 1 and 3 then columns 2 and 3 swapped and column 2 doubled
C	$= \frac{27}{2}\sqrt{6}$ square units	**a** $\begin{pmatrix} 5^4 & -2(5^4) \\ -2(5^4) & 4(5^4) \end{pmatrix}$ **b** $5^{n-1}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$

<table>
<tr><td></td><td>Yates' correction</td></tr>
<tr><td>1</td><td>$$X^2_{Yates} = \sum \frac{(|O_i - E_i| - 0.5)^2}{E_i}$$
Apply when $\nu = 1$ (i.e. a 2 x 2 table)</td></tr>
<tr><td>2</td><td>
a
<table>
<tr><th>E_i</th><th>Fish 'n' chips</th><th>Salad</th></tr>
<tr><td>Year 12</td><td>16.2</td><td>14.8</td></tr>
<tr><td>Year 13</td><td>16.8</td><td>15.2</td></tr>
</table>
b
<table>
<tr><th>E_i</th><th>Fish 'n' chips</th><th>Salad</th></tr>
<tr><td>Year 12</td><td>0.32</td><td>0.35</td></tr>
<tr><td>Year 13</td><td>0.31</td><td>0.34</td></tr>
</table>
The X^2 contributions (3 sf) have been calculated using Yates' correction.
The test statistic is 1.30 (3 sf)
The sum of the individual contributions.
c The test statistic is less than the critical value. There is insufficient evidence to reject the null hypothesis.
</td></tr>
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