Summer Independent Learning:

A level Further Mathematics

Welcome to A level further maths!

This SIL has 8 sections. Each section should take roughly 60-90 minutes:

- Sections 1-3 are **optional** preparation. You should complete these (or parts of these) if you need to revise surds and quadratics from GCSE.
- Sections 4-7 are **compulsory**. They introduce one of the most important topics in A level further maths: complex numbers.
- Section 8 is **optional**. It ties together sections 1-7 and takes them a little bit further to introduce another further maths topic.

For each section:

- Click the link to watch the video.
- Practice the skills until you are confident by answering as many of the questions in the section as you feel you need to.
- Check your answers.
- Correct any mistakes in a different colour and try to write down why you got the wrong answer.
- Write down any questions you still need to ask.

For your first further maths lesson:

- Bring an A4 ringbinder specifically for further maths.
- Bring all the summer work you have completed.
- Be ready to ask your questions and help answer each other's questions.

Aims of the summer work:

Further maths A level is about a lot more than simply getting the correct answer. It requires you to question what you already know about maths and try to look at the abstract structures. Throughout this work you will explore quadratics – in particular, the roots of quadratics. You will learn what mathematicians do when they find a quadratic that cannot be solved; or rather, one that cannot be solved using only real numbers…

Section 1: Simplifying Surds

Video: https://youtu.be/M2xDgnetfJ8

1 Do not use your calculator for this exercise. Simplify:

a $\sqrt{28}$ **b** $\sqrt{72}$ **c** $\sqrt{50}$

d $\sqrt{32}$ **e** $\sqrt{90}$ **f** $\frac{\sqrt{12}}{2}$

g $\frac{\sqrt{27}}{3}$ **h** $\sqrt{20}+\sqrt{80}$ **i** $\sqrt{200}+\sqrt{18}-\sqrt{72}$

j $\sqrt{175}+\sqrt{63}+2\sqrt{28}$ **k** $\sqrt{28}-2\sqrt{63}+\sqrt{7}$ **l** $\sqrt{80}-2\sqrt{20}+3\sqrt{45}$

m $3\sqrt{80}-2\sqrt{20}+5\sqrt{45}$ **n** $\frac{\sqrt{44}}{\sqrt{11}}$ **o** $\sqrt{12}+3\sqrt{48}+\sqrt{75}$

2 Expand and simplify if possible:

a $\sqrt{3}(2+\sqrt{3})$ **b** $\sqrt{5}(3-\sqrt{3})$ **c** $\sqrt{2}(4-\sqrt{5})$

d $(2-\sqrt{2})(3+\sqrt{5})$ **e** $(2-\sqrt{3})(3-\sqrt{7})$ **f** $(4+\sqrt{5})(2+\sqrt{5})$

g $(5-\sqrt{3})(1-\sqrt{3})$ **h** $(4+\sqrt{3})(2-\sqrt{3})$ **i** $(7-\sqrt{11})(2+\sqrt{11})$

3 Simplify $\sqrt{75}-\sqrt{12}$ giving your answer in the form $a\sqrt{3}$, where a is an integer.

Answers to Questions

Section 1: Simplifying Surds

1
- **a** $2\sqrt{7}$
- **b** $6\sqrt{2}$
- **c** $5\sqrt{2}$
- **d** $4\sqrt{2}$
- **e** $3\sqrt{10}$
- **f** $\sqrt{3}$
- **g** $\sqrt{3}$
- **h** $6\sqrt{5}$
- **i** $7\sqrt{2}$
- **j** $12\sqrt{7}$
- **k** $-3\sqrt{7}$
- **l** $9\sqrt{5}$
- **m** $23\sqrt{5}$
- **n** 2
- **o** $19\sqrt{3}$

2
- **a** $2\sqrt{3}+3$
- **b** $3\sqrt{5}-\sqrt{15}$
- **c** $4\sqrt{2}-\sqrt{10}$
- **d** $6+2\sqrt{5}-3\sqrt{2}-\sqrt{10}$
- **e** $6-2\sqrt{7}-3\sqrt{3}+\sqrt{21}$
- **f** $13+6\sqrt{5}$
- **g** $8-6\sqrt{3}$
- **h** $5-2\sqrt{3}$
- **i** $3+5\sqrt{11}$

3 $3\sqrt{3}$

Section 2: Rationalising the Denominator of Surds

Video: https://youtu.be/qiUbYB86eU0

1 Simplify:

a $\frac{1}{\sqrt{5}}$ **b** $\frac{1}{\sqrt{11}}$ **c** $\frac{1}{\sqrt{2}}$ **d** $\frac{\sqrt{3}}{\sqrt{15}}$

e $\frac{\sqrt{12}}{\sqrt{48}}$ **f** $\frac{\sqrt{5}}{\sqrt{80}}$ **g** $\frac{\sqrt{12}}{\sqrt{156}}$ **h** $\frac{\sqrt{7}}{\sqrt{63}}$

2 Rationalise the denominators and simplify:

a $\frac{1}{1+\sqrt{3}}$ **b** $\frac{1}{2+\sqrt{5}}$ **c** $\frac{1}{3-\sqrt{7}}$ **d** $\frac{4}{3-\sqrt{5}}$ **e** $\frac{1}{\sqrt{5}-\sqrt{3}}$

f $\frac{3-\sqrt{2}}{4-\sqrt{5}}$ **g** $\frac{5}{2+\sqrt{5}}$ **h** $\frac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **i** $\frac{11}{3+\sqrt{11}}$ **j** $\frac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

k $\frac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **l** $\frac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **m** $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

3 Rationalise the denominators and simplify:

a $\frac{1}{(3-\sqrt{2})^2}$ **b** $\frac{1}{(2+\sqrt{5})^2}$ **c** $\frac{4}{(3-\sqrt{2})^2}$

d $\frac{3}{(5+\sqrt{2})^2}$ **e** $\frac{1}{(5+\sqrt{2})(3-\sqrt{2})}$ **f** $\frac{2}{(5-\sqrt{3})(2+\sqrt{3})}$

Challenge

a Simplify $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.

b Hence show that $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{24}+\sqrt{25}}=4$

Answers to Questions

Section 2: Rationalising the Denominator of Surds

1 **a** $\frac{\sqrt{5}}{5}$ **b** $\frac{\sqrt{11}}{11}$ **c** $\frac{\sqrt{2}}{2}$

d $\frac{\sqrt{5}}{5}$ **e** $\frac{1}{2}$ **f** $\frac{1}{4}$

g $\frac{\sqrt{13}}{13}$ **h** $\frac{1}{3}$

2 **a** $\frac{1-\sqrt{3}}{-2}$ **b** $\sqrt{5}-2$ **c** $\frac{3+\sqrt{7}}{2}$

d $3+\sqrt{5}$ **e** $\frac{\sqrt{5}+\sqrt{3}}{2}$ **f** $\frac{(3-\sqrt{2})(4+\sqrt{5})}{11}$

g $5(\sqrt{5}-2)$ **h** $5(4+\sqrt{14})$ **i** $\frac{11(3-\sqrt{11})}{-2}$

j $\frac{5-\sqrt{21}}{-2}$ **k** $\frac{14-\sqrt{187}}{3}$ **l** $\frac{35+\sqrt{1189}}{6}$

m -1

3 **a** $\frac{11+6\sqrt{2}}{49}$ **b** $9-4\sqrt{5}$ **c** $\frac{44+24\sqrt{2}}{49}$

d $\frac{81-30\sqrt{2}}{529}$ **e** $\frac{13+2\sqrt{2}}{161}$ **f** $\frac{7-3\sqrt{3}}{11}$

Challenge

a $a-b$

b $\frac{(\sqrt{1}-\sqrt{2})+(\sqrt{2}-\sqrt{3})+\ldots+(\sqrt{24}-\sqrt{25})}{-1}=\sqrt{25}-\sqrt{1}=4$

Section 3: Methods for Solving Quadratics

Video: https://youtu.be/UikL2lyGLrU

Choose a method to solve each of the following quadratic equations. Once you have found the two roots can you then add them to find the sum and multiply them to find the product? There is a connection between these two values and the original quadratic.

	Quadratic Equation	Method	Roots	Sum of roots	Product of roots
A	$x^2 + 8x + 12 = 0$				
B	$x^2 + 4x - 15 = 0$				
C	$x^2 - 9x - 1 = 0$				
D	$2x^2 + 5x + 2$				
E	$x^2 + 9x - 11 = 0$				
F	$2x^2 + 12x - 1 = 0$				
G	$3x^2 + x - 2 = 0$				
H	$-x^2 - 4x + 5 = 0$				
I	$4x^2 - 9 = 0$				

Answers to Questions

Section 3: Methods for Solving Quadratics

	Quadratic Equation	Method	Roots	Sum of roots	Product of roots
A	$x^2+8x+12=0$	Factorise	$-6\ and -2$	-8	12
B	$x^2+4x-15=0$	Complete the square	$-2\pm\sqrt{19}$	-4	-15
C	$x^2-9x-1=0$	Complete the square/formula	$\frac{9\pm\sqrt{85}}{2}$	9	-1
D	$2x^2+5x+2$	Factorise	$-\frac{1}{2}\ and -2$	$-\frac{5}{2}$	1
E	$x^2+9x-11=0$	Complete the square/formula	$\frac{-9\pm5\sqrt{5}}{2}$	-9	-11
F	$2x^2+12x-1=0$	Complete the square/formula	$-3\pm\frac{1}{2}\sqrt{38}$	-6	$-\frac{1}{2}$
G	$3x^2+x-2=0$	Factorise	$-1\ and \frac{2}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$
H	$-x^2-4x+5=0$	Factorise	$1\ and -5$	-4	-5
I	$4x^2-9=0$	Factorise	$\pm\frac{3}{2}$	0	$-\frac{9}{4}$

Section 4: Intro to Complex Numbers

Video: https://youtu.be/dIJ6MhE3OyQ

1 Write each of the following in the form bi where b is a real number.

a $\sqrt{-9}$ **b** $\sqrt{-49}$ **c** $\sqrt{-121}$ **d** $\sqrt{-10000}$ **e** $\sqrt{-225}$

f $\sqrt{-5}$ **g** $\sqrt{-12}$ **h** $\sqrt{-45}$ **i** $\sqrt{-200}$ **j** $\sqrt{-147}$

2 Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

a $(5 + 2i) + (8 + 9i)$ **b** $(4 + 10i) + (1 - 8i)$

c $(7 + 6i) + (-3 - 5i)$ **d** $\left(\frac{1}{2} + \frac{1}{3}i\right) + \left(\frac{5}{2} + \frac{5}{3}i\right)$

e $(20 + 12i) - (11 + 3i)$ **f** $(2 - i) - (-5 + 3i)$

g $(-4 - 6i) - (-8 - 8i)$ **h** $(3\sqrt{2} + i) - (\sqrt{2} - i)$

i $(-2 - 7i) + (1 + 3i) - (-12 + i)$ **j** $(18 + 5i) - (15 - 2i) - (3 + 7i)$

3 Simplify, giving your answers in the form $a + bi$, where $a, b \in \mathbb{R}$.

a $2(7 + 2i)$ **b** $3(8 - 4i)$

c $2(3 + i) + 3(2 + i)$ **d** $5(4 + 3i) - 4(-1 + 2i)$

e $\frac{6 - 4i}{2}$ **f** $\frac{15 + 25i}{5}$

g $\frac{9 + 11i}{3}$ **h** $\frac{-8 + 3i}{4} - \frac{7 - 2i}{2}$

4 Write in the form $a + bi$, where a and b are simplified surds.

a $\frac{4 - 2i}{\sqrt{2}}$ **b** $\frac{2 - 6i}{1 + \sqrt{3}}$

5 Given that $z = 7 - 6i$ and $w = 7 + 6i$, find,
in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $z - w$ **b** $w + z$

6 Given that $z_1 = a + 9i$, $z_2 = -3 + bi$ and $z_2 - z_1 = 7 + 2i$, find a and b where $a, b \in \mathbb{R}$.

7 Given that $z_1 = 4 + i$ and $z_2 = 7 - 3i$, find, in the form $a + bi$, where $a, b \in \mathbb{R}$:

a $z_1 - z_2$ **b** $4z_2$ **c** $2z_1 + 5z_2$

8 Given that $z = a + bi$ and $w = a - bi$, $a, b \in \mathbb{R}$, show that:

a $z + w$ is always real **b** $z - w$ is always imaginary

Answers to Questions

Section 4: Intro to Complex Numbers

1	**a**	$3i$	**b**	$7i$	**c**	$11i$	**d**	$100i$
	e	$15i$	**f**	$i\sqrt{5}$	**g**	$2i\sqrt{3}$	**h**	$3i\sqrt{5}$
	i	$10i\sqrt{2}$	**j**	$7i\sqrt{3}$				
2	**a**	$13 + 11i$	**b**	$5 + 2i$	**c**	$4 + i$	**d**	$3 + 2i$
	e	$9 + 9i$	**f**	$7 - 4i$	**g**	$4 + 2i$	**h**	$2\sqrt{2} + 2i$
	i	$11 - 5i$	**j**	0				
3	**a**	$14 + 4i$	**b**	$24 - 12i$	**c**	$12 + 5i$	**d**	$24 + 7i$
	e	$3 - 2i$	**f**	$3 + 5i$	**g**	$3 + \frac{11}{3}i$	**h**	$-\frac{11}{2} + \frac{7}{4}i$

4 **a** $2\sqrt{2} - i\sqrt{2}$ **b** $(-1 + \sqrt{3}) + (3 - 3\sqrt{3})i$

5 **a** $-12i$ **b** 14

6 $a = -10,\ b = 11$

7 **a** $-3 + 4i$ **b** $28 - 12i$ **c** $43 - 13i$

8 **a** $z + w = (a + bi) + (a - bi) = 2a$

b $z - w = (a + bi) - (a - bi) = 2bi$

Section 5: Solving Quadratics with No Real Roots

Video: https://youtu.be/vBjd4z9gS18

1 Solve each of the following equations. Write your answers in the form $\pm bi$.

a $z^2 + 121 = 0$ **b** $z^2 + 40 = 0$ **c** $2z^2 + 120 = 0$

d $3z^2 + 150 = 38 - z^2$ **e** $z^2 + 30 = -3z^2 - 66$ **f** $6z^2 + 1 = 2z^2$

2 Solve each of the following equations.
Write your answers in the form $a \pm bi$.

a $(z - 3)^2 - 9 = -16$

b $2(z - 7)^2 + 30 = 6$

c $16(z + 1)^2 + 11 = 2$

3 Solve each of the following equations. Write your answers in the form $a \pm bi$.

a $z^2 + 2z + 5 = 0$ **b** $z^2 - 2z + 10 = 0$ **c** $z^2 + 4z + 29 = 0$

d $z^2 + 10z + 26 = 0$ **e** $z^2 + 5z + 25 = 0$ **f** $z^2 + 3z + 5 = 0$

Answers to Questions

Section 5: Solving Quadratics with No Real Roots

1 a $z = \pm 11\text{i}$ b $z = \pm 2\text{i}\sqrt{10}$ c $z = \pm 2\text{i}\sqrt{15}$

d $z = \pm 2\text{i}\sqrt{7}$ e $z = \pm 2\text{i}\sqrt{6}$ f $z = \pm \frac{1}{2}\text{i}$

2 a $z = 3 \pm \text{i}\sqrt{7}$ b $z = 7 \pm 2\text{i}\sqrt{3}$ c $z = -1 \pm \frac{3}{4}\text{i}$

3 a $z = -1 \pm 2\text{i}$ b $z = 1 \pm 3\text{i}$ c $z = -2 \pm 5\text{i}$

d $z = -5 \pm \text{i}$ e $z = -\frac{5}{2} \pm \frac{5\text{i}\sqrt{3}}{2}$ f $z = \frac{-3 \pm \text{i}\sqrt{11}}{2}$

Section 6: Multiplying Complex Numbers

Video: https://youtu.be/_MXkucyKEok

1 Simplify each of the following, giving your answers in the form $a + bi$.

a $(5 + i)(3 + 4i)$ **b** $(6 + 3i)(7 + 2i)$ **c** $(5 - 2i)(1 + 5i)$

d $(13 - 3i)(2 - 8i)$ **e** $(-3 - i)(4 + 7i)$ **f** $(8 + 5i)^2$

g $(2 - 9i)^2$ **h** $(1 + i)(2 + i)(3 + i)$

i $(3 - 2i)(5 + i)(4 - 2i)$ **j** $(2 + 3i)^3$

2 **a** Simplify $(4 + 5i)(4 - 5i)$, giving your answer in the form $a + bi$.

b Simplify $(7 - 2i)(7 + 2i)$, giving your answer in the form $a + bi$.

c Comment on your answers to parts **a** and **b**.

d Prove that $(a + bi)(a - bi)$ is a real number for any real numbers a and b.

3 Given that $(a + 3i)(1 + bi) = 25 - 39i$, find two possible pairs of values for a and b.

4 Write each of the following in its simplest form.

a i^6 **b** $(3i)^4$ **c** $i^5 + i$ **d** $(4i)^3 - 4i^3$

5 Express $(1 + i)^6$ in the form $a - bi$, where a and b are integers to be found.

6 Find the value of the real part of $(3 - 2i)^4$.

7 $f(z) = 2z^2 - z + 8$

Find: **a** $f(2i)$ **b** $f(3 - 6i)$

8 $f(z) = z^2 - 2z + 17$

Show that $z = 1 - 4i$ is a solution to $f(z) = 0$.

9 **a** Given that $i^1 = i$ and $i^2 = -1$, write i^3 and i^4 in their simplest forms.

b Write i^5, i^6, i^7 and i^8 in their simplest forms.

c Write down the value of:

i i^{100} **ii** i^{253} **iii** i^{301}

Answers to Questions

Section 6: Multiplying Complex Numbers

1 **a** $11 + 23i$ **b** $36 + 33i$ **c** $15 + 23i$ **d** $2 - 110i$
e $-5 - 25i$ **f** $39 + 80i$ **g** $-77 - 36i$ **h** $10i$
i $54 - 62i$ **j** $-46 + 9i$

2 **a** 41 **b** 53 **c** They are both real
d $(a + bi)(a - bi) = a^2 + b^2$, which is real.

3 $a = 7, b = -6$ or $a = 18, b = -\frac{7}{3}$

4 **a** -1 **b** 81 **c** $2i$ **d** $-60i$

5 $-8i, a = 0, b = -8$

6 $-119 - 120i$, so real part is -119

7 **a** $-2i$ **b** $-49 - 66i$

8 Substitute $z = 1 - 4i$ into $f(z)$ to get $f(z) = 0$.

9 **a** $i^3 = -i, i^4 = 1$ **b** $i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1$
c **i** 1 **ii** i **iii** i

Section 7: Dividing Complex Numbers

Video: https://youtu.be/7H0tFv7_96g

1 Write down the complex conjugate z^* for:

a $z = 8 + 2\text{i}$ **b** $z = 6 - 5\text{i}$ **c** $z = \frac{2}{3} - \frac{1}{2}\text{i}$ **d** $z = \sqrt{5} + \text{i}\sqrt{10}$

2 Find $z + z^*$ and zz^* for:

a $z = 6 - 3\text{i}$ **b** $z = 10 + 5\text{i}$ **c** $z = \frac{3}{4} + \frac{1}{4}\text{i}$ **d** $z = \sqrt{5} - 3\text{i}\sqrt{5}$

3 Write each of the following in the form $a + b\text{i}$.

a $\dfrac{3 - 5\text{i}}{1 + 3\text{i}}$ **b** $\dfrac{3 + 5\text{i}}{6 - 8\text{i}}$ **c** $\dfrac{28 - 3\text{i}}{1 - \text{i}}$ **d** $\dfrac{2 + \text{i}}{1 + 4\text{i}}$

4 Write $\dfrac{(3 - 4\text{i})^2}{1 + \text{i}}$ in the form $x + \text{i}y$ where $x, y \in \mathbb{R}$.

5 Given that $z_1 = 1 + \text{i}$, $z_2 = 2 + \text{i}$ and $z_3 = 3 + \text{i}$, write each of the following in the form $a + b\text{i}$.

a $\dfrac{z_1 z_2}{z_3}$ **b** $\dfrac{(z_2)^2}{z_1}$ **c** $\dfrac{2z_1 + 5z_3}{z_2}$

6 Given that $\dfrac{5 + 2\text{i}}{z} = 2 - \text{i}$, find z in the form $a + b\text{i}$.

Answers to Questions

Section 7: Dividing Complex Numbers

1 **a** $8 - 2\text{i}$ **b** $6 + 5\text{i}$ **c** $\frac{2}{3} + \frac{1}{2}\text{i}$ **d** $\sqrt{5} - \text{i}\sqrt{10}$

2 **a** $z + z^* = 12,\ zz^* = 45$ **b** $z + z^* = 20,\ zz^* = 125$

c $z + z^* = \frac{3}{2},\ zz^* = \frac{5}{8}$ **d** $z + z^* = 2\sqrt{5},\ zz^* = 50$

3 **a** $-\frac{6}{5} - \frac{7}{5}\text{i}$ **b** $-\frac{11}{50} + \frac{27}{50}\text{i}$ **c** $\frac{31}{2} + \frac{25}{2}\text{i}$ **d** $\frac{6}{17} - \frac{7}{17}\text{i}$

4 $-\frac{31}{2} - \frac{17}{2}\text{i}$

5 **a** $\frac{3}{5} + \frac{4}{5}\text{i}$ **b** $\frac{7}{2} + \frac{1}{2}\text{i}$ **c** $\frac{41}{5} - \frac{3}{5}\text{i}$

6 $\frac{8}{5} + \frac{9}{5}\text{i}$

Section 8: Roots and Coefficients of Quadratics

Video: https://youtu.be/dwzyOBSHk8w

1 The roots of the quadratic equation $z^2 + 2z + 26 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

2 The roots of the quadratic equation $z^2 - 8z + 25 = 0$ are α and β.

Find: **a** α and β **b** $\alpha + \beta$ **c** $\alpha\beta$

3 Given that 2 + 3i is one of the roots of a quadratic equation with real coefficients,

a write down the other root of the equation

b find the quadratic equation, giving your answer in the form $z^2 + bz + c = 0$ where b and c are real constants.

4 Given that 5 − i is a root of the equation $z^2 + pz + q = 0$, where p and q are real constants,

a write down the other root of the equation

b find the value of p and the value of q.

5 Given that $z_1 = -5 + 4i$ is one of the roots of the quadratic equation $z^2 + bz + c = 0$, where b and c are real constants, find the values of b and c.

6 Given that 1 + 2i is one of the roots of a quadratic equation with real coefficients, find the equation giving your answer in the form $z^2 + bz + c = 0$ where b and c are integers to be found.

Answers to Questions

Section 8: Roots and Coefficients of Quadratics

1 **a** $-1 + 5i, -1 - 5i$ **b** -2 **c** 26

2 **a** $4 + 3i, 4 - 3i$ **b** 8 **c** 25

3 **a** $2 - 3i$ **b** $z^2 - 4z + 13 = 0$

4 **a** $5 + i$

b $(z - (5 - i))(z - (5 + i)) = 0$

$z^2 - (5 + i)z - (5 - i)z + (5 - i)(5 + i) = 0$

$z^2 - 10z + 26 = 0 \Rightarrow p = -10, q = 26$

5 $z^2 + 10z + 41 = 0$

6 $z^2 - 2z + 5 = 0$